

GRADED q -SCHUR ALGEBRAS

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ABSTRACT. Generalizing recent work of Brundan and Kleshchev, we introduce grading on Dipper-James' q -Schur algebra, and prove a graded analogue of the Leclerc and Thibon's conjecture on the decomposition numbers of the q -Schur algebra.

1. INTRODUCTION

In [15], Khovanov and Lauda introduced generalization of the degenerate affine nilHecke algebra of type A, in order to categorify $U_v^-(\mathfrak{g})$, the negative half of the quantized enveloping algebra associated with a simply-laced quiver. The algebra is called the *Khovanov-Lauda algebra*. They also proposed the study of cyclotomic Khovanov-Lauda algebras. Soon after that, Brundan and Kleshchev proved in [4] that the cyclotomic Khovanov-Lauda algebras associated with a cyclic quiver are nothing but block algebras of the cyclotomic Hecke algebras of type $G(m, 1, n)$ and, more recently, they proved the graded analogue of an old result of the author of this note [3] in [5]. The aim of this note is to introduce grading on the q -Schur algebra and obtain the graded analogue of the decomposition number conjecture for the q -Schur algebra considered in [23].

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2. PRELIMINARIES I; THE HECKE ALGEBRA

Let F be a field, $q \in F^\times$ a primitive e^{th} root of unity where $e \geq 2$. The *Hecke algebra* of type A, which we denote by \mathcal{H}_n , is the F -algebra defined by generators T_1, \dots, T_{n-1} and relations

$$(T_i - q)(T_i + 1) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \text{ (if } j \neq i \pm 1\text{)}.$$

As the Artin braid relations hold, we have well-defined elements T_w , for $w \in S_n$, and they form an F -basis of \mathcal{H}_n . We also have pairwise commuting

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elements X_1, \dots, X_n which are defined by $X_1 = 1$, $X_{k+1} = q^{-1}T_k X_k T_k$, for $1 \leq k < n$. They are invertible in \mathcal{H}_n .

The Hecke algebra \mathcal{H}_n has the F -algebra automorphism Ψ of order 2 which is defined by $T_i \mapsto q - 1 - T_i$. It sends T_w to $(-q)^{\ell(w)}(T_{w^{-1}})^{-1}$, for $w \in S_n$.

The Hecke algebra \mathcal{H}_n also has the anti- F -algebra automorphism of order 2 that fixes the generators T_i . It sends T_w to $T_w^* := T_{w^{-1}}$, for $w \in S_n$.

Let $I = \mathbb{Z}/e\mathbb{Z}$, and $\underline{i} = (i_1, \dots, i_n) \in I^n$. We call \underline{i} a *residue sequence*. The symmetric group S_n acts on I^n by place permutation. That is,

$$w\underline{i} = (i_{w^{-1}(1)}, \dots, i_{w^{-1}(n)}), \text{ for } w \in S_n.$$

We denote by s_k the transposition of k and $k+1$. Thus,

$$s_k \underline{i} = (i_1, \dots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \dots, i_n).$$

By the Specht module theory, there are primitive central idempotents $e(\underline{i})$ of the F -subalgebra of \mathcal{H}_n generated by X_1, \dots, X_n . The idempotent $e(\underline{i})$ corresponds to the simultaneous eigenvalue

$$(X_1, \dots, X_n) \mapsto (q^{i_1}, \dots, q^{i_n}).$$

Thus, we have $\sum_{\underline{i} \in I^n} e(\underline{i}) = 1$ and $e(\underline{i})e(\underline{j}) = \delta_{\underline{i}, \underline{j}} e(\underline{i})$, for $\underline{i}, \underline{j} \in I^n$. Note that $e(\underline{i})$ may be zero, and it is nonzero only when it comes from the residue sequence of a standard λ -tableau, for some $\lambda \vdash n$.¹ In particular, we always have $i_1 = 0 \pmod{e}$ whenever $e(\underline{i}) \neq 0$.

Brundan and Kleshchev introduced the following elements t_1, \dots, t_n and $\sigma_1, \dots, \sigma_{n-1}$ in [4]. The definition of t_a is easy to state and it is

$$t_a = \sum_{\underline{i} \in I^n} (1 - q^{-i_a} X_a) e(\underline{i}).$$

Note that $t_1 = 0$ by the remark above. Then, [4, Lemma 2.1] implies that t_2, \dots, t_n are nilpotent.² The definition of $\sigma_1, \dots, \sigma_{n-1}$ is more involved. They introduce Laurent series $P_k(\underline{i})$ and $Q_k(\underline{i})$ in t_1, \dots, t_n as follows.

$$P_k(\underline{i}) = \begin{cases} 1 & (\text{if } i_{k+1} = i_k), \\ (1 - q)(1 - q^{i_k - i_{k+1}}(1 - t_k)(1 - t_{k+1})^{-1})^{-1} & (\text{if } i_{k+1} \neq i_k), \end{cases}$$

$$Q_k(\underline{i}) = \begin{cases} 1 - q + qt_{k+1} - t_k & (\text{if } i_{k+1} = i_k), \\ q^{i_k} & (\text{if } i_{k+1} = i_k - 1), \\ \frac{q^{i_k}(1 - t_k) - q^{i_{k+1}+1}(1 - t_{k+1})}{(q^{i_k}(1 - t_k) - q^{i_{k+1}+1}(1 - t_{k+1}))^2} & (\text{if } i_{k+1} = i_k + 1), \\ \frac{q^{i_k}(1 - t_k) - q^{i_{k+1}+1}(1 - t_{k+1})}{q^{i_k}(1 - t_k) - q^{i_{k+1}+1}(1 - t_{k+1})} & (\text{if } i_{k+1} \neq i_k \pm 1). \end{cases}$$

¹Recall that if k is located on the a_k^{th} row and the b_k^{th} column of a standard λ -tableau, the residue sequence associated with the tableau is defined by $i_k = -a_k + b_k \pmod{e}$, for $1 \leq k \leq n$.

²Dr. Lauda informed the author that he and Alex Hoffnig determined upperbound for the degree of nilpotency for cyclotomic Hecke algebras, and it implies $t_a = 0$, for $1 \leq a \leq e - 1$, in our case.

Then these Laurent series well-define elements in \mathcal{H}_n by the nilpotency, and we define

$$\sigma_k = \sum_{\underline{i} \in I^n} (T_k + P_k(\underline{i})) Q_k^{-1}(\underline{i}) e(\underline{i}).$$

The main result of [4] stated in our case is the following.³ As we will need assume $e \geq 4$ in later sections, we exclude the case $e = 2$ in the following theorem. When $e = 2$, the last two relations in the theorem must be modified. See [4, Main Theorem] for the details. Note that the theorem allows us to view \mathcal{H}_n as a \mathbb{Z} -graded F -algebra. We define

$$\deg(e(\underline{i})) = 0, \quad \deg(t_a) = 2, \quad \deg(\sigma_k e(\underline{i})) = \begin{cases} -2 & (\text{if } i_k = i_{k+1}), \\ 1 & (\text{if } i_k - i_{k+1} = \pm 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Theorem 2.1. *Suppose that $e \geq 3$. Then \mathcal{H}_n is defined by three sets of generators*

$$\begin{cases} e(\underline{i}), & \text{for } \underline{i} \in I^n \text{ such that } i_1 = 0, \\ t_1, t_2, \dots, t_n, & \text{where } t_1 = 0, \\ \sigma_1, \dots, \sigma_{n-1} \end{cases}$$

and relations

$$\begin{aligned} \sum_{\underline{i} \in I^n} e(\underline{i}) &= 1, \quad e(\underline{i})e(\underline{j}) = \delta_{\underline{i}, \underline{j}} e(\underline{i}), \\ t_a t_b &= t_b t_a, \quad t_a e(\underline{i}) = e(\underline{i}) t_a, \quad \sigma_k e(\underline{i}) = e(s_k \underline{i}) \sigma_k, \\ \sigma_k t_a &= t_a \sigma_k \text{ if } a \neq k, k+1, \\ \sigma_k t_{k+1} - t_k \sigma_k &= t_{k+1} \sigma_k - \sigma_k t_k = \sum_{i_k = i_{k+1}} e(\underline{i}), \\ \sigma_k \sigma_l &= \sigma_l \sigma_k \text{ if } l \geq k+2, \\ \sigma_k^2 &= \sum_{i_k - i_{k+1} \neq 0, \pm 1} e(\underline{i}) + \sum_{i_k - i_{k+1} = 1} (t_k - t_{k+1}) e(\underline{i}) + \sum_{i_k - i_{k+1} = -1} (t_{k+1} - t_k) e(\underline{i}), \\ \sigma_k \sigma_{k+1} \sigma_k - \sigma_{k+1} \sigma_k \sigma_{k+1} &= \sum_{i_{k+2} = i_k = i_{k+1} - 1} e(\underline{i}) - \sum_{i_{k+2} = i_k = i_{k+1} + 1} e(\underline{i}). \end{aligned}$$

Example 2.2. Suppose that $e \geq 3$ as above. Define $\underline{i}_{\pm} = (0, \pm 1) \in I^2$. Then, \mathcal{H}_2 has the F -basis $e(\underline{i}_{\pm})$ and $t_1 = t_2 = \sigma_1 = 0$.

The new generators are called the *Khovanov-Lauda generators*. As the Hecke algebra \mathcal{H}_n is a graded F -algebra now, we may consider the category of finite dimensional graded \mathcal{H}_n -modules. Here, we require homomorphisms

³In fact, the set of relations stated in [loc. cit], which are the Khovanov-Lauda relations, is slightly weaker, and hence their assertion is slightly stronger: we may deduce $t_1 = 0$ and $e(\underline{i}) = 0$ whenever $i_1 \neq 0$, from the Khovanov-Lauda relations.

to be degree preserving. We denote this category by $\mathcal{H}_n\text{-mod}^{\mathbb{Z}}$. We denote the obvious forgetful functor by

$$\text{For} : \mathcal{H}_n\text{-mod}^{\mathbb{Z}} \rightarrow \mathcal{H}_n\text{-mod}.$$

Let $A = \bigoplus_{k \in \mathbb{Z}} A_k$ be a finite dimensional graded F -algebra over a field F . We adopt the following convention throughout the paper.

Definition 2.3. An A -module M is a *graded right A -module* if it is a \mathbb{Z} -graded vector space

$$M = \bigoplus_{l \in \mathbb{Z}} M_l$$

such that $M_l A_k \subseteq M_{l+k}$.

An A -module M is a *graded left A -module* if it is a \mathbb{Z} -graded vector space $M = \bigoplus_{l \in \mathbb{Z}} M_l$ such that $A_k M_l \subseteq M_{l-k}$.

For right and left modules, the shift functor in $A\text{-mod}^{\mathbb{Z}}$ is defined by $M[1]_k = M_{k+1}$, for $k \in \mathbb{Z}$.

For a graded left A -module M , we denote

$$M^\circ = \bigoplus_{k \in \mathbb{Z}} (M^\circ)_k \quad \text{and} \quad (M^\circ)_k = \text{Hom}_F(M_k, F),$$

which is a graded right A -module in the natural way. We call M° the *natural dual* of M .

The following basic facts on graded algebras will be used frequently in the rest of the paper without further notice.

Theorem 2.4. *Let A be a finite dimensional graded F -algebra over a field F , $A\text{-mod}^{\mathbb{Z}}$ the category of finite dimensional graded right A -modules, and $\text{For} : A\text{-mod}^{\mathbb{Z}} \rightarrow A\text{-mod}$ the forgetful functor.*

- (a) *A graded A -module X is indecomposable if and only if $\text{For}(X)$ is indecomposable.*
- (b) *Let X and Y be indecomposable. Then $\text{For}(X) \simeq \text{For}(Y)$ if and only if $X \simeq Y[k]$, for some $k \in \mathbb{Z}$.*

Proof. See [13, Theorem 3.2] for (a) and [13, Theorem 4.1] for (b). □

We have $e(\underline{i})^* = e(\underline{i})$, $t_a^* = t_a$ but $\sigma_k^* \neq \sigma_k$. However, as we will do later, we may use cellular algebra structure of \mathcal{H}_n to define graded dual Specht modules. For the involution Ψ , we have

$$e(\underline{i}) \mapsto e(-\underline{i}) \quad \text{and} \quad t_a \mapsto \sum_{\underline{i} \in I^n} (1 - t_a)^{-1} e(-\underline{i}),$$

but $\Psi(\sigma_k) \neq \sigma_k$. To remedy this, we define

$$e(\underline{i})' = \Psi(e(\underline{i})), \quad t_a' = \Psi(t_a), \quad \sigma_k' = \Psi(\sigma_k).$$

Then, we may use these elements as new Khovanov-Lauda generators and make \mathcal{H}_n into a graded F -algebra. We denote this graded Hecke algebra by \mathcal{H}'_n . Thus, we have the isomorphism of graded F -algebras

$$\Psi : \mathcal{H}_n \simeq \mathcal{H}'_n,$$

by $e(\underline{i}) \mapsto e(\underline{i})'$, $t_a \mapsto t'_a$, $\sigma_k \mapsto \sigma'_k$. Later, we will introduce graded Specht modules S'^{λ} for \mathcal{H}'_n by using this isomorphism.

Remark 2.5. To study the graded module theory for \mathcal{H}_n , we have to introduce another anti-involution as follows.

Definition 2.6. The anti- F -algebra automorphism of \mathcal{H}_n of order 2 which fixes the Khovanov-Lauda generators is denoted by $h \mapsto h^\sharp$. Thus,

$$e(\underline{i})^\sharp = e(\underline{i}), \quad t_a^\sharp = t_a, \quad \sigma_k^\sharp = \sigma_k$$

and $(h_1 h_2)^\sharp = h_2^\sharp h_1^\sharp$, for $h_1, h_2 \in \mathcal{H}_n$.

Definition 2.7. Let $M = \bigoplus_{k \in \mathbb{Z}} M_k$ be a graded \mathcal{H}_n -module. Then, the *dual* of M is the graded \mathcal{H}_n -module $M^\sharp = \bigoplus_{k \in \mathbb{Z}} M_k^\sharp$, where

$$M_k^\sharp = \bigoplus_{k \in \mathbb{Z}} (M^\sharp)_k \quad \text{and} \quad (M^\sharp)_k = \text{Hom}_F(M_{-k}, F),$$

such that the module structure is given by $fh(m) = f(mh^\sharp)$, for $f \in M^\sharp$, $h \in \mathcal{H}_n$ and $m \in M$.

Hence, if a matrix representation of M in the Khovanov-Lauda generators is given by $e(\underline{i}) \mapsto \hat{e}(\underline{i})$, $t_a \mapsto \hat{t}_a$, $\sigma_k \mapsto \hat{\sigma}_k$, then the transposed matrices

$$e(\underline{i}) \mapsto {}^t \hat{e}(\underline{i}), \quad t_a \mapsto {}^t \hat{t}_a, \quad \sigma_k \mapsto {}^t \hat{\sigma}_k$$

gives a matrix representation of M^\sharp . Thus, we have

$$T_k \mapsto \sum ({}^t \hat{\sigma}_k \hat{Q}_k(\underline{i}) - \hat{P}_k(\underline{i}) {}^t \hat{e}(\underline{i})),$$

where $\hat{P}_k(\underline{i})$ and $\hat{Q}_k(\underline{i})$ are obtained from $P_k(\underline{i})$ and $Q_k(\underline{i})$ by replacing t_a with ${}^t \hat{t}_a$.

Remark 2.8. Introduce a filtration

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n(n-1)/2} = \mathcal{H}_n$$

on \mathcal{H}_n by declaring that F_ℓ , for $0 \leq \ell \leq n(n-1)/2$, is the F -span of the products of generators $e(\underline{i})$, t_a and σ_k such that $\sigma_1, \dots, \sigma_{n-1}$ appear in the product at most ℓ times in total. Define $\text{gr}^F \mathcal{H}_n$ to be the associated graded F -algebra. We denote the image of σ_k in $\text{gr}^F \mathcal{H}_n$ by $\bar{\sigma}_k$. Then, we may well-define $\bar{\sigma}_w$, for $w \in S_n$, in $\text{gr}^F \mathcal{H}_n$ because the Artin braid relations hold in $\text{gr}^F \mathcal{H}_n$. We remark that $\{\bar{\sigma}_w \mid w \in S_n\}$ is not an F -basis of $\text{gr}^F \mathcal{H}_n$. We can only say that the elements $t_1^{a_1} \cdots t_n^{a_n} e(\underline{i}) \bar{\sigma}_w$, for $a_1, \dots, a_n \geq 0$, $\underline{i} \in I^n$ and $w \in S_n$, span $\text{gr}^F \mathcal{H}_n$. Many of them are zero. This fact will cause a problem when we try to make the permutation modules of \mathcal{H}_n into graded

modules, and we will appeal to a result by Hemmer and Nakano [14] to bypass this difficulty.

For each $w \in S_n$, we choose a reduced expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ and define $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_{\ell(w)}}$, which is a lift of $\bar{\sigma}_w \in \text{gr}^F \mathcal{H}_n$. Then, we have

$$\mathcal{H}_n = \sum_{a_1, \dots, a_n \geq 0} \sum_{\underline{i} \in I^n} \sum_{w \in S_n} F t_1^{a_1} \cdots t_n^{a_n} e(\underline{i}) \sigma_w.$$

3. PRELIMINARIES II; THE q -SCHUR ALGEBRA

For partitions and compositions we follow standard notation. We denote the conjugate partition of a partition λ by λ' .

For a composition $\mu = (\mu_1, \mu_2, \dots, \mu_r) \models n$, we have the Young subgroup

$$S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_r}.$$

The number r is denoted by $\ell(\mu)$ and called the *length* or *depth* of μ . We define $x_\mu = \sum_{w \in S_\mu} T_w \in \mathcal{H}_n$. The right \mathcal{H}_n -module $M^\mu = x_\mu \mathcal{H}_n$ is called the *permutation module* associated with μ . Then, the *q -Schur algebra* is defined by

$$\mathcal{S}_{d,n} = \text{End}_{\mathcal{H}_n}(\oplus_{\mu \models n, \ell(\mu) \leq d} M^\mu).$$

Recall that, by applying the involution Ψ to x_μ , we obtain

$$y_\mu = \sum_{w \in S_\mu} (-q)^{-\ell(w)} T_w,$$

up to a nonzero scalar. The right \mathcal{H}_n -module $N^\mu = y_\mu \mathcal{H}_n$ is called the *signed permutation module* associated with μ .

By twisting the action on M^μ by Ψ , we obtain the \mathcal{H}_n -module $(M^\mu)^\Psi$. Then $(M^\mu)^\Psi \simeq N^\mu$, so that we may consider that M^μ and N^μ have the same underlying vector space. Observe that $\varphi \in \text{Hom}_F(M^\nu, M^\mu)$ belongs to $\text{Hom}_{\mathcal{H}_n}(M^\nu, M^\mu)$ if and only if it belongs to $\text{Hom}_{\mathcal{H}_n}(N^\nu, N^\mu)$. Thus,

$$\text{Hom}_{\mathcal{H}_n}(M^\nu, M^\mu) = \text{Hom}_{\mathcal{H}_n}(N^\nu, N^\mu),$$

and we have $\mathcal{S}_{d,n} = \text{End}_{\mathcal{H}_n}(\oplus_{\mu \models n, \ell(\mu) \leq d} N^\mu)$ as in [8, Theorem 2.24].

The q -Schur algebra is a factor algebra of the quantum algebra $U_q(\mathfrak{gl}_d)$ and the isomorphism classes of simple $\mathcal{S}_{d,n}$ -modules are given by highest weight theory. We denote by $L(\lambda)$ the simple $\mathcal{S}_{d,n}$ -module associated with a highest weight, or a partition, $\lambda \vdash n$.

Recall also that the category $\mathcal{S}_{d,n}\text{-mod}$ is a highest weight category whose standard and costandard modules are given by Weyl modules $W(\lambda)$ and Schur modules $H^0(\lambda)$, respectively. The category has tilting modules $T(\lambda)$, which is the indecomposable $\mathcal{S}_{d,n}$ -module with the property that

- (1) there is a monomorphism $W(\lambda) \rightarrow T(\lambda)$ in $\mathcal{S}_{d,n}\text{-mod}$ such that the cokernel has Weyl filtration which uses only $W(\mu)$, for $\mu \triangleleft \lambda$,

- (2) there is an epimorphism $T(\lambda) \rightarrow H^0(\lambda)$ in $\mathcal{S}_{d,n}$ -mod such that the kernel has Schur filtration⁴ which uses only $H^0(\mu)$, for $\mu \triangleleft \lambda$.

We suppose $d \geq n$ throughout the paper. Thus, we have the projector

$$e : \oplus_{\mu \models n, \ell(\mu) \leq d} M^\mu \rightarrow M^{(1^n)}.$$

It is an idempotent in $\mathcal{S}_{d,n}$. Then, we have $\mathcal{H}_n \simeq e\mathcal{S}_{d,n}e$. The isomorphism is given by sending $h \in \mathcal{H}_n$ to $\varphi_h \in \text{Hom}_{\mathcal{H}_n}(M^{(1^n)}, M^{(1^n)}) = e\mathcal{S}_{d,n}e$, for $h \in \mathcal{H}_n$, where φ_h is defined by $m \mapsto hm$, for $m \in M^{(1^n)}$. For our purposes, we should view this Hecke algebra as \mathcal{H}'_n without grading.

The functor $\mathcal{S}_{d,n}\text{-mod} \rightarrow \mathcal{H}_n\text{-mod}$ given by $M \mapsto Me = M \otimes_{\mathcal{S}_{d,n}} \mathcal{S}_{d,n}e$ is called the *Schur functor*.

Remark 3.1. If we view $e \in \mathcal{S}_{d,n}$ as the projector

$$e : \oplus_{\mu \models n, \ell(\mu) \leq d} N^\mu \rightarrow N^{(1^n)},$$

we obtain the similar isomorphism $\mathcal{H}_n \simeq e\mathcal{S}_{d,n}e$, which sends $h \in \mathcal{H}_n$ to $\varphi_h \in \text{Hom}_{\mathcal{H}_n}(N^{(1^n)}, N^{(1^n)}) = e\mathcal{S}_{d,n}e$, for $h \in \mathcal{H}_n$. We view this Hecke algebra as \mathcal{H}_n without grading. Then Ψ gives the isomorphism

$$\mathcal{H}_n \simeq e\mathcal{S}_{d,n}e \simeq \mathcal{H}'_n.$$

The $(\mathcal{S}_{d,n}, \mathcal{H}_n)$ -bimodule structure and $(\mathcal{S}_{d,n}, \mathcal{H}'_n)$ -bimodule structure on $\mathcal{S}_{d,n}e$ give two Schur functors

$$\mathcal{S}_{d,n}\text{-mod} \rightarrow \mathcal{H}_n\text{-mod} \quad \text{and} \quad \mathcal{S}_{d,n}\text{-mod} \rightarrow \mathcal{H}'_n\text{-mod}.$$

The image of the first Schur functor is obtained from the image of the second by twisting the action by Ψ .

Remark 3.2. The q -Schur algebra has the anti-involution $*$ which restricts to the $*$ on the Hecke algebra, and we may consider the dual $M^* = \text{Hom}_F(M, F)$ of a $\mathcal{S}_{d,n}$ -module M . The Schur functor commutes with taking duals [9, p.83 Remarks], and $H^0(\lambda) \simeq W(\lambda)^*$, for $\lambda \vdash n$, by [9, Proposition 4.1.6].

Remark 3.3. Let $P(\lambda)$ and $I(\lambda)$ be the projective cover and the injective envelope of a simple $\mathcal{S}_{d,n}$ -module $L(\lambda)$, for $\lambda \vdash n$, respectively. Then, $P(\lambda)^* \simeq I(\lambda)$ [9, 4.3], and both $P(\lambda)$ and $I(\lambda)$ map to a self-dual \mathcal{H}_n -module called the Young module associated with λ . Later, we will introduce graded Young modules \tilde{Y}'^λ for \mathcal{H}'_n and graded signed Young modules Y^λ for \mathcal{H}_n . If we forget the grading, we have $\tilde{Y}'^\lambda \simeq (Y^{\lambda'})^\Psi$. They are self-dual. That is, $(\tilde{Y}'^\lambda)^* \simeq \tilde{Y}'^\lambda$ and $(Y^\lambda)^* \simeq Y^\lambda$ if we forget the grading.

Let \mathbf{t}^μ be the *canonical tableau* associated with $\mu \models n$: \mathbf{t}^μ is the row standard μ -tableau such that $1, \dots, \mu_1$ are in the first row, $\mu_1 + 1, \dots, \mu_1 + \mu_2$ are in the second row, etc. Then, a row standard tableau \mathbf{t} defines an element $d(\mathbf{t}) \in S_n$: if k is the (a_k, b_k) -entry of \mathbf{t}^μ then $d(\mathbf{t})(k)$ is the (a_k, b_k) -entry of \mathbf{t} , for $1 \leq k \leq n$. The element $d(\mathbf{t})$ is the distinguished coset representative in $S_\mu d(\mathbf{t})$.

⁴It is usually called good filtration.

Definition 3.4. Let \mathbf{s} and \mathbf{t} be row standard μ -tableaux. Then we define

$$m_{\mathbf{st}} = T_{d(\mathbf{s})}^* x_\mu T_{d(\mathbf{t})}.$$

Murphy showed that these elements for standard μ -tableaux \mathbf{s} and \mathbf{t} for partitions $\mu \vdash n$ form a cellular basis of \mathcal{H}_n [19, 3.20]. Later, we will introduce graded dual Specht modules based on this fact.

Recall that a *tableau of weight ν* is a tableau with ν_1 1's, ν_2 2's, etc. as entries.

Definition 3.5. Let $\lambda \vdash n$ and $\nu \models n$. For a semistandard λ -tableau S of weight ν , we define $\nu^{-1}(S)$ to be the set of standard λ -tableaux \mathbf{s} such that if we replace $1, \dots, \nu_1$ by $1, \nu_1 + 1, \dots, \nu_1 + \nu_2$ by 2, etc. then we obtain S .

Definition 3.6. Let $\lambda \vdash n$, and $\mu \models n, \nu \models n$. For a semistandard λ -tableau S of weight μ and a semistandard λ -tableau T of weight ν , we define

$$m_{ST} = \sum_{\mathbf{s} \in \nu^{-1}(S)} \sum_{\mathbf{t} \in \nu^{-1}(T)} m_{\mathbf{st}}.$$

In particular, if T is a standard λ -tableau \mathbf{t} we have the element $m_{S, \mathbf{t}}$.

Theorem 3.7. *The elements $m_{S, \mathbf{t}}$, for semistandard λ -tableaux S of weight μ , where λ runs through all partitions of n , form a basis of M^μ .*

See [19, Theorem 4.9] for the proof. Next we recall two ways to give a basis of $\text{Hom}_{\mathcal{H}_n}(M^\nu, M^\mu)$. See [19, Theorem 4.7] and [19, Theorem 4.13]. The latter gives a cellular basis of $\mathcal{S}_{d, n}$. Later, we will introduce graded Schur modules based on this fact.

Theorem 3.8. *For distinguished coset representatives $d \in S_\mu \backslash S_n / S_\nu$, define $\varphi_{\mu\nu}^d \in \text{Hom}_{\mathcal{H}_n}(M^\nu, M^\mu)$ by $x_\nu \mapsto \sum_{w \in S_\mu d S_\nu} T_w$. Then they form a basis of $\text{Hom}_{\mathcal{H}_n}(M^\nu, M^\mu)$.*

Theorem 3.9. *Define $\varphi_{ST} \in \text{Hom}_{\mathcal{H}_n}(M^\nu, M^\mu)$ by $x_\nu \mapsto m_{ST}$. Then these elements, where λ runs through all partitions of n , form a basis of $\text{Hom}_{\mathcal{H}_n}(M^\nu, M^\mu)$.*

We want to make the q -Schur algebra into a graded F -algebra. As $m_{S, \mathbf{t}}$ form a basis of M^μ by Theorem 3.7, it is natural to expect that replacing $T_{d(\mathbf{t})}$ with $\sigma_{d(\mathbf{t})}$ in the definition of $m_{S, \mathbf{t}}$, for row standard μ -tableaux \mathbf{t} , would give a homogeneous basis of M^μ , which then would allow us to grade M^μ and $\mathcal{S}_{d, n}$. However, this is not the case even in the \mathcal{H}_2 case, as $\sigma_1 = 0$ there. \mathcal{H}_2 has $e(\underline{i}_\pm)$ as a basis, so that we have to consider a basis of M^μ obtained by not only using σ_w but also using other Khovanov-Lauda generators. This is not easy to control in general.

Example 3.10. Let \underline{i}_\pm be as in Example 2.2. Then, the basis elements $e(\underline{i}_\pm)$ act on permutation modules as follows.

$$m_{(2)} e(\underline{i}_+) = m_{(2)}, \quad m_{(2)} e(\underline{i}_-) = 0,$$

and

$$m_{(1^2)}e(\underline{i}_+) = \frac{1}{q+1}m_{(1^2)}, \quad m_{(1^2)}e(\underline{i}_-) = m_{(1^2)} - \frac{1}{q+1}m_{(2)}.$$

A right approach is to grade signed Young modules. Then, we may grade the signed permutation modules N^μ by using decomposition into direct sum of signed Young modules, so that we have grading on $\mathcal{S}_{d,n}$.

Before proceeding further, we recall the main result of [6]. It says that the first idea which failed for the permutation modules M^μ works for Specht modules S^λ , and we obtain graded Specht modules. The difference from the permutation modules is the fact that S^λ is generated by the element z_λ , whose definition will be given below, and that z_λ is a simultaneous eigenvector of X_1, \dots, X_n .

Let $\mathcal{N}^{\triangleright\lambda}$ be the F -span of the elements $m_{\mathbf{st}}$ where \mathbf{s} and \mathbf{t} are standard μ -tableaux for some $\mu \triangleright \lambda$. It is well-known that $\mathcal{N}^{\triangleright\lambda}$ is a two-sided ideal of \mathcal{H}_n . Define the element z_λ by

$$z_\lambda = x_\lambda + \mathcal{N}^{\triangleright\lambda} \in \mathcal{H}_n / \mathcal{N}^{\triangleright\lambda}.$$

The *Specht module* associated with λ is the right \mathcal{H}_n -module $S^\lambda = z_\lambda \mathcal{H}_n$. We also denote it by S_r^λ when we stress that it is a right module. As we already said, z_λ is a simultaneous eigenvector of X_1, \dots, X_n , which implies that $S^\lambda = \sum_{w \in S_n} F z_\lambda \sigma_w$.

Remark 3.11. Note that the Dipper-James' Specht module in [7], which is identified with Donkin's Specht module $Sp(\lambda)$ in [9, Proposition 4.5.8], is $(S^{\lambda'})^\Psi$, which is the dual $(S^\lambda)^*$ of S^λ by [9, Proposition 4.5.9]. If λ is e -restricted then $D^\lambda = S^\lambda / \text{Rad } S^\lambda$ is the simple \mathcal{H}_n -module which is the image of $L(\lambda)$ under the Schur functor. We call the Dipper-James' Specht modules *dual Specht modules*.

We consider the graded Hecke algebra \mathcal{H}_n and introduce graded Specht modules for \mathcal{H}_n .

We already know that $m_{\mathbf{t}} = z_\lambda T_{d(\mathbf{t})}$, for standard λ -tableaux \mathbf{t} , form a basis of the Specht module. We fix a reduced expression for each $w \in S_n$, and define

$$v_{\mathbf{t}} = z_\lambda \sigma_{d(\mathbf{t})}.$$

For a standard tableau \mathbf{t} , denote by x_k , for $1 \leq k \leq n$, the node occupied with k , and $\lambda_{\mathbf{t}}(k)$ the partition which consists of x_1, \dots, x_k . We view x_k as a removable node of $\lambda_{\mathbf{t}}(k)$. We define $N_{\mathbf{t}}^b(k)$ to be the number of addable $\text{res}(x_k)$ -nodes of $\lambda_{\mathbf{t}}(k)$ which is strictly below x_k minus the number of removable $\text{res}(x_k)$ -nodes of $\lambda_{\mathbf{t}}(k)$ which is strictly below x_k , for $1 \leq k \leq n$. Then we declare that $v_{\mathbf{t}}$ is homogenous of degree

$$\deg(v_{\mathbf{t}}) = \sum_{k=1}^n N_{\mathbf{t}}^b(k).$$

The homogeneous basis depends on the choice of reduced expressions of $d(\mathbf{t})$, but the grading on the Specht module defined by the grading of the

homogeneous basis does not. This grading is compatible with the grading on \mathcal{H}_n . Hence, the Specht modules are made into graded \mathcal{H}_n -modules. See [6, Theorem 4.10] for the details of these statements.

Definition 3.12. We denote by S^λ the graded \mathcal{H}_n -module defined above and call it the *graded Specht module* for \mathcal{H}_n associated with $\lambda \vdash n$.

Next we introduce graded dual Specht modules. To do this, first we work with left modules instead of right modules and define the graded Specht module S_l^λ for \mathcal{H}_n in the same way. The fact that the Murphy basis $m_{\mathbf{st}}$ form a cellular basis with respect to the anti-involution $*$ implies that the dual Specht module $(S^\lambda)^*$ is isomorphic to the right module $(S_l^\lambda)^\circ$ when we ignore grading.⁵ Hence, it is natural to define as follows.

Definition 3.13. Let S_l^λ be the left graded Specht module for \mathcal{H}_n . Then we define the *graded dual Specht module* \tilde{S}^λ , for $\lambda \vdash n$, by

$$\tilde{S}^\lambda = (S_l^\lambda)^\circ \in \mathcal{H}_n\text{-mod}^{\mathbb{Z}}.$$

Remark 3.14. Let $v_{\mathbf{t}} = z_\lambda \sigma_{d(\mathbf{t})}$ be the basis of S^λ . For each basis element $v_{\mathbf{t}}$, we define

$$\tilde{v}_{\mathbf{t}} = (\sigma_{d(\mathbf{t})})^\# z_\lambda.$$

Then the dual basis of $\tilde{v}_{\mathbf{t}}$ form a basis of \tilde{S}^λ . The representing matrix of σ_k on \tilde{S}^λ with respect to the basis is not given by simply transposing the representing matrix of σ_k on S^λ with respect to the basis $v_{\mathbf{t}}$. If we transpose the representing matrices of the Khovanov-Lauda generators, we obtain $(S^\lambda)^\#$ instead of \tilde{S}^λ .

Next we introduce graded Specht modules for \mathcal{H}'_n .

Definition 3.15. We define the *graded Specht module* S'^λ , for $\lambda \vdash n$, by

$$S'^\lambda = (\tilde{S}^{\lambda'})^\Psi \in \mathcal{H}'_n\text{-mod}^{\mathbb{Z}}.$$

As we are given graded Specht modules S^λ , our next task is to grade signed Young modules for \mathcal{H}_n by using them as building blocks. In fact, we first work with left modules and construct left graded signed Young modules Y_l^λ by using S_l^λ . Then we define graded signed Young modules for \mathcal{H}_n . To do this, we must assume that $e \geq 4$. Hence,

from now on, we assume that $e \geq 4$ and $d \geq n$.

In [14, 4.3], the authors explain how to construct Young modules in the way similar to construction of tilting modules for quasi-hereditary algebras. As the Specht modules they use are dual Specht modules in our notation,

⁵Consider the matrix representation of S^λ with respect to the basis $z_\lambda T_{d(\mathbf{t})}$ and apply $*$ to find that the representing matrix of T_i on S_l^λ with respect to the basis $T_{d(\mathbf{t})}^* z_\lambda$ is the transpose of the representing matrix of T_i on S^λ , which is the representing matrix of T_i on $(S_l^\lambda)^\circ$ as well as $(S^\lambda)^*$.

we apply the involution Ψ everywhere and transpose partitions everywhere in [loc. cit]. Note that, by the natural transformation

$$\mathrm{Hom}(\mathrm{For}(S_l^\lambda), \mathrm{For}(-)) \simeq \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}(S_l^\lambda[k], -),$$

we have $\mathrm{Ext}^1(\mathrm{For}(S_l^\lambda), \mathrm{For}(-)) \simeq \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^1(S_l^\lambda[k], -)$.

Let $\lambda^{[0]} = \lambda$ and $W_0 = S_l^\lambda$. Suppose that we have already constructed partitions $\lambda^{[0]}, \dots, \lambda^{[s]}$ and graded \mathcal{H}_n -modules W_0, \dots, W_s . Then we choose $\lambda^{[s+1]}$ maximal with respect to the dominance order such that

$$a_{s+1} := \sum_{k \in \mathbb{Z}} a_{s+1}[k] > 0, \text{ where } a_{s+1}[k] = \dim_F \mathrm{Ext}^1(S_l^{\lambda^{[s+1]}}[k], W_s).$$

Then we define W_{s+1} by the corresponding short exact sequence

$$0 \rightarrow W_s \rightarrow W_{s+1} \rightarrow \bigoplus_{k \in \mathbb{Z}} (S_l^{\lambda^{[s+1]}}[k])^{\oplus a_{s+1}[k]} \rightarrow 0.$$

Note that $\lambda^{[s+1]} \triangleleft \lambda^{[t]}$, for some $t \leq s$. Otherwise, $\mathrm{Ext}^1(S_l^{\lambda^{[s+1]}}[k], S_l^{\lambda^{[t]}}) = 0$, for all $k \in \mathbb{Z}$ and all $t \leq s$, by [14, Proposition 4.2.1], so that it implied $a_{s+1} = 0$. As the poset of partitions $\lambda \vdash n$ is finite, the process must terminate after finitely many steps. We denote the resulting module W_N , for the terminal N , by Y_l^λ . Note that we have $\mathrm{Ext}^1(S_l^\mu[k], Y_l^\lambda) = 0$, for all $k \in \mathbb{Z}$ and for all $\mu \vdash n$.

We call Y_l^λ the *left graded signed Young module* for \mathcal{H}_n associated with λ . Then we define as follows.

Definition 3.16. The *graded signed Young module* for \mathcal{H}_n is defined by

$$Y^\lambda = (Y_l^\lambda)^\circ.$$

This is justified by the self-duality of the signed Young modules in the non-graded case and the following result, which follows from [14, Theorem 4.6.2].

Proposition 3.17. *For (Y^λ) is the signed Young module which is the image of the tilting $\mathcal{S}_{d,n}$ -module $T(\lambda)$ under the Schur functor with respect to the $(\mathcal{H}_n, \mathcal{S}_{d,n})$ -bimodule structure.*

Next we introduce graded Young modules for \mathcal{H}'_n .

Definition 3.18. The *graded Young module* for \mathcal{H}'_n is defined by

$$\tilde{Y}'^\lambda = (Y^{\lambda'})^\Psi.$$

The following is clear by the relationship between Young modules and the signed Young modules.

Proposition 3.19. *For (\tilde{Y}'^λ) is the Young module which is the image of the indecomposable projective $\mathcal{S}_{d,n}$ -module $P(\lambda)$ under the Schur functor with respect to the $(\mathcal{H}'_n, \mathcal{S}_{d,n})$ -bimodule structure.*

We are ready to grade signed permutation modules for \mathcal{H}_n . Recall that the signed Young modules $For(Y^\lambda)$, for $\lambda \vdash n$, form a complete set of the isomorphism classes of indecomposable summands of $\oplus_{\mu \vdash n, \ell(\mu) \leq d} N^\mu$, by [9, 4.4]. Write N^μ as a direct sum of $For(Y^\lambda)$'s, where only λ with $\lambda \supseteq \mu'$ can appear by [9, 4.4]. By replacing $For(Y^\lambda)$ with Y^λ , we obtain the *graded signed permutation module*, which we also denote by N^μ . We have proved the following theorem.

Theorem 3.20. *Suppose that $e \geq 4$. Then the q -Schur algebra can be made into a \mathbb{Z} -graded F -algebra.*

In fact, we may replace each indecomposable direct summand $For(Y^\lambda)$ of N^μ with any shift of Y^λ . Different choice of the shifts leads to different grading on $\mathcal{S}_{d,n}$. We want that the grading on $\mathcal{S}_{d,n}$ is compatible with the grading on \mathcal{H}_n , which we now explain.

Observe that $\text{End}(\mathcal{H}_n) \simeq (\mathcal{H}_n)_0$, the degree zero part of \mathcal{H}_n . We write the identity $1 \in \mathcal{H}_n$ into sum of pairwise orthogonal primitive idempotents in $(\mathcal{H}_n)_0$. Let f be one of the primitive idempotents. Then, $f\mathcal{H}_n \simeq Y^\lambda[k]$, for some $\lambda \vdash n$ and some $k \in \mathbb{Z}$. We shall replace $f\mathcal{H}_n$ with $Y^\lambda[k]$. Namely,

we choose the shifts so that $N^{(1^n)} \simeq \mathcal{H}_n$ in $\mathcal{H}_n\text{-mod}^{\mathbb{Z}}$.

Then, we also have $\mathcal{H}'_n \simeq M^{(1^n)} = (N^{(1^n)})^\Psi$ in $\mathcal{H}'_n\text{-mod}^{\mathbb{Z}}$. Hereafter, we fix a choice of the shifts satisfying this condition.

Remark 3.21. Define the *graded tensor space* by

$$V^{\otimes n} = \oplus_{\mu \vdash n, \ell(\mu) \leq d} M^\mu \in \mathcal{H}'_n\text{-mod}^{\mathbb{Z}}.$$

Then we may write

$$\mathcal{S}_{d,n} = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(V^{\otimes n}, V^{\otimes n}[k]).$$

As in the Hecke algebra case, we denote the category of finite dimensional graded $\mathcal{S}_{d,n}$ -modules by $\mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}}$.

4. GRADED SCHUR FUNCTORS

Let $e \in \mathcal{S}_{d,n}$ be the projector from $\oplus_{\mu \vdash n, \ell(\mu) \leq d} N^\mu$ to $N^{(1^n)}$. This is an idempotent and homogeneous of degree 0.

Lemma 4.1. *We have the following isomorphism of graded F -algebras.*

$$\mathcal{H}_n \simeq \bigoplus_{k \in \mathbb{Z}} \text{Hom}(N^{(1^n)}, N^{(1^n)}[k]) = e\mathcal{S}_{d,n}e.$$

Proof. We look at the signed permutation module $N^{(1^n)}$. We already know that if we ignore the grading, then

$$\mathcal{H}_n \simeq e\mathcal{S}_{d,n}e = \text{End}_{\mathcal{H}_n}(N^{(1^n)})$$

and the isomorphism is given by $h \mapsto \varphi_h$, the left multiplication by $h \in \mathcal{H}_n$. As $y_{(1^n)} = 1$, the multiplication by a homogeneous element of degree

k , for $k \in \mathbb{Z}$, gives an endomorphism of degree k . To see this, we write the identity into the sum of pairwise orthogonal primitive idempotents in $(\mathcal{H}_n)_0$ as before. Let f and f' be two of the primitive idempotents. Since $N^{(1^n)} \simeq \mathcal{H}_n$ as graded \mathcal{H}_n -modules, we may consider f and f' as degree zero elements of $N^{(1^n)}$. Let $h \in f'\mathcal{H}_nf$ be homogeneous of degree k . Then $f'h \in (N^{(1^n)})_k$. Thus, $hf = f'h$ implies that the left multiplication by h gives $f\mathcal{H}_n \rightarrow f'\mathcal{H}_n[k]$. We have the isomorphism

$$\mathcal{H}_n \simeq \bigoplus_{k \in \mathbb{Z}} \text{Hom}(N^{(1^n)}, N^{(1^n)}[k]) = e\mathcal{S}_{d,n}e$$

of graded F -algebras. \square

As a corollary, we have another isomorphism of graded F -algebras

$$\mathcal{H}'_n \simeq \bigoplus_{k \in \mathbb{Z}} \text{Hom}(M^{(1^n)}, M^{(1^n)}[k]) = e\mathcal{S}_{d,n}e.$$

Using the $(\mathcal{S}_{d,n}, \mathcal{H}_n)$ -bimodule structure of $\mathcal{S}_{d,n}e$, we define

$$\mathcal{F}_1 = - \otimes_{\mathcal{S}_{d,n}} \mathcal{S}_{d,n}e : \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}} \rightarrow \mathcal{H}_n\text{-mod}^{\mathbb{Z}}.$$

The degree k part of $\mathcal{F}_1(M)$, for $M \in \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}}$, is $M_k e$. We call the functor \mathcal{F}_1 the *graded Schur functor of the first kind*. The right adjoint functor is defined as follows.

$$\mathcal{G}_1 = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(\mathcal{S}_{d,n}e[-k], -) : \mathcal{H}_n\text{-mod}^{\mathbb{Z}} \rightarrow \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}}.$$

Thus, the degree k part of $\mathcal{G}_1(M)$ is $\text{Hom}(\mathcal{S}_{d,n}e, M[k])$ and $\mathcal{F}_1 \circ \mathcal{G}_1(M) \simeq M$, for $M \in \mathcal{H}_n\text{-mod}^{\mathbb{Z}}$.

Next we use the $(\mathcal{S}_{d,n}, \mathcal{H}'_n)$ -bimodule structure to define

$$\mathcal{F}_2 = - \otimes_{\mathcal{S}_{d,n}} \mathcal{S}_{d,n}e : \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}} \rightarrow \mathcal{H}'_n\text{-mod}^{\mathbb{Z}}$$

and the right adjoint functor

$$\mathcal{G}_2 = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(\mathcal{S}_{d,n}e[-k], -) : \mathcal{H}'_n\text{-mod}^{\mathbb{Z}} \rightarrow \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}}.$$

We call the functor \mathcal{F}_2 the *graded Schur functor of the second kind*. We have $\mathcal{F}_2 \circ \mathcal{G}_2(M) \simeq M$, for $M \in \mathcal{H}'_n\text{-mod}^{\mathbb{Z}}$.

The following definition is justified by [14, Theorem 3.4.2].

Definition 4.2. We define the *graded Weyl module* by $W(\lambda) = \mathcal{G}_1(S^\lambda)$, and the *graded tilting module* by $T(\lambda) = \mathcal{G}_1(Y^\lambda)$.

Remark 4.3. If we follow the recipe for constructing tilting modules, we obtain certain shift of $T(\lambda)$, and it is rather complicated to determine the shift.

Next we introduce Schur modules. As in the Hecke algebra case, first we work with the left modules instead of right modules and introduce the graded Weyl module $W_l(\lambda)$, for $\lambda \vdash n$, in the same way. Then, as the

semistandard basis defined by m_{ST} form a cellular basis with respect to the anti-involution $*$, the Schur module is isomorphic to $(W_l(\lambda))^\circ$ when we ignore grading. Hence, we define as follows.

Definition 4.4. Let $W_l(\lambda)$ be the left graded Weyl module. Define the *graded Schur modules* $H^0(\lambda)$, for $\lambda \vdash n$, by

$$H^0(\lambda) = (W_l(\lambda))^\circ.$$

Graded simple modules are defined as follows.

Definition 4.5. Define $D^\lambda = S^\lambda / \text{Rad } S^\lambda$ if λ is e -restricted, and define $L(\lambda) = W(\lambda) / \text{Rad } W(\lambda)$ for all $\lambda \vdash n$.

By [13, Proposition 3.5], $\text{Rad } S^\lambda$ (resp. $\text{Rad } W(\lambda)$) is a graded submodule of S^λ (resp. $W(\lambda)$), so that D^λ and $L(\lambda)$ are graded simple \mathcal{H}_n -modules and graded simple $\mathcal{S}_{d,n}$ -modules, respectively.

We record the following theorem [5, Theorem 5.10]. It says that D^λ is self-dual not only with respect to the usual non-graded anti-involution $*$ in $\mathcal{H}_n\text{-mod}$ but also with respect to the graded anti-involution \sharp in $\mathcal{H}_n\text{-mod}^{\mathbb{Z}}$.

Theorem 4.6. *The graded simple \mathcal{H}_n -module D^λ is self-dual. Namely, we have $(D^\lambda)^\sharp \simeq D^\lambda$.*

We define another sets of graded Weyl modules, graded Schur modules and graded simple modules by

$$W'(\lambda) = \mathcal{G}_2(S'^\lambda), \quad H'^0(\lambda) = (W'_l(\lambda))^\circ, \quad L'(\lambda) = W'(\lambda) / \text{Rad } W'(\lambda).$$

Definition 4.7. We denote the projective cover of $L'(\lambda)$ by $P'(\lambda)$.

Lemma 4.8. *We have $P'(\lambda) = \mathcal{G}_2(\tilde{Y}'^\lambda)$, for $\lambda \vdash n$.*

Proof. As we have a monomorphism $S_l^{\lambda'} \rightarrow Y_l^{\lambda'}$, we have the epimorphism

$$\tilde{Y}'^\lambda = ((Y_l^{\lambda'})^\circ)^\Psi \rightarrow ((S_l^{\lambda'})^\circ)^\Psi = S'^\lambda.$$

Thus, by [14, Theorem 3.3.4(ii)], we have the epimorphism

$$\mathcal{G}_2(\tilde{Y}'^\lambda) \rightarrow \mathcal{G}_2(S'^\lambda) = W'(\lambda).$$

We have $\mathcal{G}_2(\tilde{Y}'^\lambda) \simeq P'(\lambda)[k]$, for some $k \in \mathbb{Z}$, by [14, Corollary 3.8.2]. Thus, the existence of the epimorphism $P'(\lambda)[k] \rightarrow W'(\lambda)$ implies $k = 0$. \square

The following is our main object of study.

Definition 4.9. The *graded decomposition number* $d_{\lambda\mu}(v)$, for $\lambda \vdash n$ and $\mu \vdash n$, is the Laurent polynomial defined by

$$d_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (W(\lambda) : L(\mu)[k]) v^k,$$

where $(W(\lambda) : L(\mu)[k])$ is the multiplicity of $L(\mu)[k]$ in the composition factors of $W(\lambda)$.

Note that $L(\lambda)[k]$, for $\lambda \vdash n$ and $k \in \mathbb{Z}$, form a complete set of graded simple $\mathcal{S}_{d,n}$ -modules. If μ is e -restricted, $L(\mu) = \mathcal{G}_1(D^\mu)$, so that we have

$$d_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (S^\lambda : D^\mu[k]) v^k.$$

Using the second sets of graded Weyl modules and graded simple modules, we define another graded decomposition number $d'_{\lambda\mu}(v)$ by

$$d'_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (W'(\lambda) : L'(\mu)[k]) v^k.$$

Lemma 4.10. *We denote $S_\lambda = S_l^{\lambda'}$ and $D_\mu = \text{Soc } S_l^{\mu'}$ if μ is e -restricted. Then we have*

$$d'_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (S_\lambda : D_\mu[k]) v^k = d_{\lambda\mu}(v).$$

Proof. Let $D'^\mu = S'^\mu / \text{Rad } S'^\mu$ and recall that $S'^\lambda = ((S_l^{\lambda'})^\circ)^\Psi$. Then

$$(W'(\lambda) : L'(\mu)[k]) = (S'^\lambda : D'^\mu[k]) = (S_l^{\lambda'} : \text{Soc } S_l^{\mu'}[k]).$$

Thus, the first equality follows.

For the second equality, observe that $((S_\lambda)^\circ)^\Psi = S^\lambda$. □

5. GRADED DECOMPOSITION NUMBERS

Here, we recall the Leclerc-Thibon basis of the Fock space. Let Λ be the ring of symmetric functions with coefficients in $\mathbb{Q}(v)$, and let s_λ be the Schur polynomial associated with a partition λ . Each node x of λ has the residue $\text{res}(x)$: if it is on the a^{th} row and the b^{th} column of λ , then $\text{res}(x) = -a + b \in \mathbb{Z}/e\mathbb{Z}$. The quantized enveloping algebra U_v of type $A_{e-1}^{(1)}$, which is generated by the Chevalley generators e_i 's f_i 's and the Cartan torus part, acts on Λ by

$$e_i s_\lambda = \sum_{\text{res}(\lambda/\mu)=i} v^{-N_i^a(\lambda/\mu)} s_\mu, \quad f_i s_\lambda = \sum_{\text{res}(\mu/\lambda)=i} v^{N_i^b(\mu/\lambda)} s_\mu,$$

for $i \in \mathbb{Z}/e\mathbb{Z}$, etc. where $N_i^a(x)$ (resp. $N_i^b(x)$) is the number of addable i -nodes minus the number of removable i -nodes above (resp. below) x . This is called the (level 1) *deformed Fock space*. To identify our Fock space with those used in [17] and [23], transpose partitions.

In [17], the authors introduced the bar-involution on the deformed Fock space, and defined two kinds of the canonical bases on Λ . One of the basis, which consists of the elements b_μ^+ , for partitions μ , is characterized by the bar-invariance and the triangularity with requirement about polynomiality:

$$\overline{b_\mu^+} = b_\mu^+ \quad \text{and} \quad b_\mu^+ \in s_\mu + \sum_{\lambda \triangleright \mu} v\mathbb{Z}[v] s_\lambda.$$

Let $\mu = (\mu_1, \dots, \mu_r)$ be a partition, and consider the infinite sequence

$$(i_1, \dots, i_r, i_{r+1}, \dots) = (\mu_1, \mu_2 - 1, \dots, \mu_r - r + 1, -r, -r - 1, \dots).$$

In the Fermionic description of the deformed Fock space, s_μ is the infinite wedge $u_{i_1} \wedge u_{i_2} \wedge \cdots$. Let A_r be the number of pairs (i, j) with $1 \leq i < j \leq r$ and $j - i \notin e\mathbb{Z}$. Then, the bar-involution is defined by $\overline{v} = v^{-1}$ and

$$\overline{u_{i_1} \wedge \cdots \wedge u_{i_r} \wedge u_{i_{r+1}} \wedge \cdots} = (-1)^{r(r-1)/2} v^{A_r} u_{i_r} \wedge \cdots \wedge u_{i_1} \wedge u_{i_{r+1}} \wedge \cdots.$$

We do not explain the straightening law, which is explained in [17], but it is clear that if $\mu = (n)$ then $b_\mu^+ = s_\mu$.

By [17, Theorem 3.2], $\{b_\mu^+ \mid \mu \text{ is } e\text{-restricted}\}$ is the canonical basis i.e. the lower global basis of the U_v -submodule generated by the empty partition, which is isomorphic to the basic representation $V(\Lambda_0)$.

We write $b_\mu^+ = \sum_{\lambda \supseteq \mu} e_{\lambda\mu}^+(v) s_\lambda$. Define $d_{\lambda\mu} = [\text{For}(W(\lambda)) : \text{For}(L(\mu))]$.

The following was conjectured by Leclerc and Thibon in [loc. cit] and proved by Varagnolo and Vasserot [23, Theorem 11].

Theorem 5.1. *If the characteristic of F is zero, then $d_{\lambda\mu} = e_{\lambda\mu}^+(1)$, for $\lambda \vdash n$ and $\mu \vdash n$.*

In fact, as is pointed out by Leclerc in [16], who proved the q -Schur algebra analogue of the result [12, Theorem 2.4], we may prove the above theorem by using the decomposition numbers of the Hecke algebra. As we already have the graded decomposition numbers of the Hecke algebra in [5, Corollary 5.15], we may prove the graded analogue of Theorem 5.1 by the argument in the proof of [16, Theorem 1]. We will come back to this point after preparing necessary materials.

Recall the direct sum of signed permutation modules $N = \bigoplus_{\mu \models n, \ell(\mu) \leq d} N^\mu$. We define $T = \mathcal{G}_1(N)$. $\text{For}(T)$ is a full tilting $\mathcal{S}_{d,n}$ -module. Then we have

$$\mathcal{S}_{d,n} = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(N, N[k]) \simeq \bigoplus_{k \in \mathbb{Z}} \text{Hom}(T, T[k]).$$

This is the Ringel dual description of the q -Schur algebra. Thus, we have the functor⁶

$$F = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(T[-k], -) : \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}} \rightarrow \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}},$$

which induces equivalence between the full subcategory of Schur filtered $\mathcal{S}_{d,n}$ -modules and the full subcategory of Weyl filtered $\mathcal{S}_{d,n}$ -modules by [22, Theorem 6].

Lemma 5.2. *We have the following.*⁷

- (a) $F(H^0(\lambda')[k]) = W'(\lambda)[k]$, for $\lambda \vdash n$ and $k \in \mathbb{Z}$.
- (b) $F(T(\lambda')) = P'(\lambda)$, for $\lambda \vdash n$.
- (c) $\text{Hom}(P'(\mu), W'(\lambda)[k]) = \text{Hom}(T(\mu'), H^0(\lambda')[k])$.

⁶Let $V \in \mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}}$, $\varphi \in \text{Hom}(T, V[k])$ and $f \in \text{Hom}(T, T[l])$. Then the composition $T \rightarrow T[l] \rightarrow V[k+l]$ is denoted by φf .

⁷Compare (a) with [9, Proposition 4.1.5].

Proof. (a) Note that, due to our degree convention, $S_l^{\lambda'}[k] = eW_l(\lambda')[k]$ implies that

$$H^0(\lambda')[k]e = (W_l(\lambda')[k])^\circ e = (S_l^{\lambda'}[k])^\circ = \tilde{S}^{\lambda'}[k].$$

Thus, $F(H^0(\lambda')[k])e = \text{Hom}(Te, H^0(\lambda')[k]e)$ is isomorphic to

$$\text{Hom}(N^{(1^n)}, \tilde{S}^{\lambda'}[k]) \simeq \text{Hom}(M^{(1^n)}, (\tilde{S}^{\lambda'})^\Psi[k]) \simeq (\tilde{S}^{\lambda'})^\Psi[k].$$

We have proved $\mathcal{F}_2(F(H^0(\lambda')[k])) \simeq S'^\lambda[k]$. Then, since $F(H^0(\lambda'))$ has Weyl filtration, we have

$$F(H^0(\lambda')[k]) \simeq \mathcal{G}_2\mathcal{F}_2(F(H^0(\lambda')[k])) \simeq W'(\lambda)[k].$$

(b) $F(T(\lambda'))e = \text{Hom}(Te, T(\lambda')e) = \text{Hom}(N^{(1^n)}, Y^{\lambda'})$ implies that

$$F(T(\lambda'))e \simeq \text{Hom}(M^{(1^n)}, (Y^{\lambda'})^\Psi) \simeq (Y^{\lambda'})^\Psi = \tilde{Y}'^\lambda.$$

We have proved $\mathcal{F}_2(F(T(\lambda'))e) \simeq \tilde{Y}'^\lambda$. The result follows by Lemma 4.8.

(c) By (a), (b) and the equivalence of the categories of Weyl filtered modules and Schur filtered modules in the non-graded case, we have

$$\text{Hom}(P'(\mu), W'(\lambda)[k]) = \text{Hom}(T(\mu'), H^0(\lambda')[k]).$$

□

Corollary 5.3. *We have the following equalities.*

$$(a) \quad d'_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (T(\mu') : W(\lambda')[k])v^{-k} = \sum_{k \in \mathbb{Z}} (T_l(\mu') : W_l(\lambda')[k])v^k.$$

$$(b) \quad \text{Hom}(W(\lambda), H^0(\mu)[k]) = \begin{cases} F & (\text{if } \lambda = \mu \text{ and } k = 0), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. Lemma 5.2 implies that $(W'(\lambda) : L'(\mu)[k])$ is equal to

$$\dim \text{Hom}(P'(\mu)[k], W'(\lambda)) = \dim \text{Hom}(T(\mu')[k], H^0(\lambda')).$$

Thus, if we define $k_\lambda \in \mathbb{Z}$ by $\text{Hom}(W(\lambda), H^0(\lambda)[k_\lambda]) \neq 0$, then we have

$$(W'(\lambda) : L'(\mu)[k]) = (T(\mu') : W(\lambda')[-k_\lambda - k]).$$

It follows that

$$d'_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (T(\mu') : W(\lambda')[k])v^{-k_\lambda - k}.$$

As $Y^{\mu'} = (Y_l^{\mu'})^\circ$ implies $T(\mu') = T_l(\mu')^\circ$, and there is a monomorphism $W_l(\mu') \rightarrow T_l(\mu')$, we have an epimorphism $T(\mu') \rightarrow H^0(\mu')$. This implies that $(T(\mu') : W(\mu')[k]) = \delta_{k0}$. Hence, we set $\lambda = \mu$ and deduce $k_\lambda = 0$. Thus, (a) and (b) follow. Note that we do not use the natural dual in the second equality in (a). Thus, the grading of \mathcal{H}_n is sign modified when we switch to left modules, so that v is replaced with v^{-1} there. □

Remark 5.4. Define a v -analogue of the Kostka numbers by

$$K_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (N^\mu : S^{\lambda'}[k]) v^{-k}.$$

Define another polynomial $n_{\nu\mu}(v)$ which is obtained by decomposing the signed permutation modules into indecomposable direct summands:

$$n_{\nu\mu}(v) = \sum_{k \in \mathbb{Z}} (N^\mu : Y^{\nu'}[k]) v^{-k}.$$

Then we have $K_{\lambda\mu}(v) = \sum_{\nu \vdash n} d_{\lambda\nu}(v) n_{\nu\mu}(v)$.

It may be interesting to compare $K_{\lambda\mu}(v)$ with the Kostka polynomials defined by the Lascoux-Schützenberger charge statistics.

In the rest of the paper, we will prove that the formula in [5] which equates $d_{\lambda\mu}(v)$ and the parabolic Kazhdan-Lusztig polynomials $e_{\lambda\mu}^+(v^{-1})$, for e -restricted μ , holds for all μ .

Definition 5.5. Let $\lambda \vdash n$ and $\mu \vdash n$. Write $\mu = \mu^{(0)} + e\mu^{(1)}$ such that $\mu^{(0)} = (\mu_1^{(0)}, \dots, \mu_d^{(0)})$ is e -restricted. Then we define $\hat{\mu} \vdash n + d(d-1)(e-1)$ and $\tilde{\lambda}, \tilde{\mu} \vdash n + d(d-1)(e-1)$ by

$$\begin{cases} \hat{\mu} = 2(e-1)\rho_d + (\mu_d^{(0)}, \dots, \mu_1^{(0)}) + e\mu^{(1)}, \\ \tilde{\lambda} = \lambda + (e-1)(d-1, \dots, d-1), \\ \tilde{\mu} = \mu + (e-1)(d-1, \dots, d-1), \end{cases}$$

where $\rho_d = (d-1, d-2, \dots, 0)$.

Proposition 5.6. *We have the following.*

(1) *For each $\mu \vdash n$, there is a unique $s \in \mathbb{Z}$ such that*

$$\text{Hom}(L(\tilde{\mu}), T(\hat{\mu})[s]) = F, \quad \text{Hom}(L(\lambda), T(\hat{\mu})[s]) = 0, \quad \text{if } \lambda \neq \tilde{\mu}.$$

(2) *Denote the value $s \in \mathbb{Z}$ in (1) by $\text{shift}(\mu)$. Then we have*

$$d_{\lambda\mu}(v) = v^{-\text{shift}(\mu)} d'_{\tilde{\lambda}\tilde{\mu}}(v^{-1}).$$

Proof. This follows from the argument in the proof of [16, Theorem 1]. Main points are that we can use [1, Proposition 5.8] and general properties of tilting modules to prove the identity, and that restrictive assumption on e in [1] was later removed in [2], so that we have no restriction on e , here. Thus, if we ignore the grading then $T(\hat{\mu})$ is the injective envelope of $L(\mu)$ in $U_q(\mathfrak{sl}_d)$ -mod. Let \det_q be the determinant representation of $U_q(\mathfrak{gl}_d)$. Then

$$W(\tilde{\mu}) = \det_q^{\otimes (e-1)(d-1)} \otimes W(\mu).$$

As a $U_q(\mathfrak{gl}_d)$ -module, $\text{Soc } T(\hat{\mu}) \simeq L(\nu)$, for some $\nu \vdash n$ such that

$$L(\nu)|_{U_q(\mathfrak{sl}_d)} \simeq L(\mu).$$

It follows that $\nu = \tilde{\mu}$ and

$$\text{Hom}_{U_q(\mathfrak{gl}_d)}(L(\tilde{\mu}), T(\hat{\mu})) = F, \quad \text{Hom}(L(\lambda), T(\hat{\mu})) = 0, \quad \text{if } \lambda \neq \tilde{\mu}.$$

Note that we are in the case $d \leq n$. Rename d by d' and take $d \geq n$. We denote by ξ the projector to the direct sum of M^μ with $\ell(\mu) \leq d'$. Then, we may identify two F -algebras

$$\mathcal{S}_{d',n} = \xi \mathcal{S}_{d,n} \xi.$$

Applying the Hom functor

$$\text{Hom}_{\mathcal{S}_{d,n}}(\mathcal{S}_{d,n}\xi, -) : \mathcal{S}_{d,n}\text{-mod} \rightarrow \mathcal{S}_{d',n}\text{-mod},$$

which sends the Weyl module to the Weyl module with the same label, and preserves irreducibility, we return to the case $d \geq n$ and obtain

$$\text{Hom}_{\mathcal{S}_{d,n}}(L(\tilde{\mu}), T(\hat{\mu})) = F, \quad \text{Hom}(L(\lambda), T(\hat{\mu})) = 0, \text{ if } \lambda \neq \tilde{\mu}.$$

in $\mathcal{S}_{d,n}\text{-mod}$. In $\mathcal{S}_{d,n}\text{-mod}^{\mathbb{Z}}$, it implies that there is a unique $s \in \mathbb{Z}$ such that $\text{Hom}(L(\tilde{\mu}), T(\hat{\mu})[s]) = F$. It is clear that $\text{Hom}(L(\lambda), T(\hat{\mu})[s]) = 0$ if $\lambda \neq \tilde{\mu}$. We have proved (1).

Denote $s = \text{shift}(\mu)$. If $L(\mu)[k]$ appears as a subquotient of $W(\lambda)$ then the monomorphism from the subquotient to $T(\hat{\mu})[s+k]$ extends to

$$W(\tilde{\lambda})|_{U_q(\mathfrak{sl}_d)} \rightarrow T(\hat{\mu})[s+k]|_{U_q(\mathfrak{sl}_d)}.$$

Grading is absent in this statement: nevertheless, due to the fact that we will recover grading at the end, this together with (1) imply that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (W(\lambda) : L(\mu)[k]) v^k &= \sum_{k \in \mathbb{Z}} \dim \text{Hom}(W(\tilde{\lambda}), T(\hat{\mu})[s+k]) v^k \\ &= \sum_{k \in \mathbb{Z}} \dim \text{Hom}(T_l(\hat{\mu})[s+k], H_l^0(\tilde{\lambda})) v^k \\ &= \sum_{k \in \mathbb{Z}} (T_l(\hat{\mu}) : W_l(\tilde{\lambda})[-s-k]) v^k. \end{aligned}$$

The last equality is by Corollary 5.3(b). Then, it follows from Corollary 5.3(a) that

$$d_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (T_l(\hat{\mu}) : W_l(\tilde{\lambda})[k]) v^{-s-k} = v^{-s} d'_{\tilde{\lambda}'\hat{\mu}'}(v^{-1}).$$

We have proved (2). □

Observe that $\hat{\mu}'$ is e -restricted. Hence Lemma 4.10 implies that

$$d'_{\tilde{\lambda}'\hat{\mu}'}(v) = d_{\tilde{\lambda}'\hat{\mu}'}(v).$$

Now, we use results from [5]. Their deformed Fock space is dual to ours. The anti-involution which fixes the Cartan torus and interchanges e_i and f_i , for $i \in \mathbb{Z}/e\mathbb{Z}$, gives the left U_v -module structure on the dual space. Their basis which consists of M_μ 's is the dual basis of our Schur polynomial basis, and their dual canonical basis in $V(\Lambda_0)^* = \text{Hom}_{\mathbb{Q}(v)}(V(\Lambda_0), \mathbb{Q}(v))$, which is denoted $\{L_\lambda \mid \lambda \text{ is } e\text{-restricted}\}$ in [loc. cit], is the dual basis of the canonical basis in $V(\Lambda_0)$. Hence, noting the definition of $[S^\mu : D^\lambda]_q$ [5, p.7]

where notation for shifting is in the opposite direction, [5, Corollary 5.15] reads

$$d_{\lambda\mu}(v) = e_{\lambda\mu}^+(v^{-1}) \text{ if } \mu \text{ is } e\text{-restricted.}$$

Hence, if the characteristic of F is zero, then

$$d_{\lambda\mu}(v) = v^{-\text{shift}(\mu)} d_{\tilde{\lambda}'\tilde{\mu}'}(v^{-1}) = v^{-\text{shift}(\mu)} e_{\tilde{\lambda}'\tilde{\mu}'}^+(v).$$

On the other hand, the following was proved in [16, Theorem 2]. We denote the affine symmetric group by \tilde{S}_d . It acts on \mathbb{Z}^d by the level e action. Let $\nu \in \mathbb{Z}^d$ be the unique weight in the \tilde{S}_d -orbit $\tilde{S}_d(\mu + \rho_d)$ that satisfies $\nu_1 \geq \dots \geq \nu_d$ and $\nu_1 - \nu_d \leq e$. The stabilizer of ν is a standard parabolic subgroup of finite order, and it has the longest element. We denote by ℓ_μ the length of the longest element.

Theorem 5.7. *The following formula holds.*

$$e_{\lambda\mu}^+(v) = v^{\frac{d(d-1)}{2} - \ell_\mu} e_{\tilde{\lambda}'\tilde{\mu}'}^+(v^{-1}).$$

The next theorem is the main result of this paper.

Theorem 5.8. *Suppose that F has characteristic zero, $q \in F^\times$ a primitive e^{th} root of unity with $e \geq 4$. Then the Dipper-James' q -Schur algebra is a \mathbb{Z} -graded F -algebra and we have*

$$d_{\lambda\mu}(v) = e_{\lambda\mu}^+(v^{-1}).$$

Proof. By the previous formulas, we have

$$d_{\lambda\mu}(v) = v^{-\text{shift}(\mu)} e_{\tilde{\lambda}'\tilde{\mu}'}^+(v) = v^{-\text{shift}(\mu) + \frac{d(d-1)}{2} - \ell_\mu} e_{\lambda\mu}^+(v^{-1}).$$

We set $\lambda = \mu$ to deduce that $\text{shift}(\mu) = \frac{d(d-1)}{2} - \ell_\mu$. □

To summarize, the Leclerc-Thibon canonical basis which consists of b_μ^+ 's computes the graded decomposition numbers of the q -Schur algebra at e^{th} roots of unity in a field of characteristic zero where $e \geq 4$.

6. EXAMPLES

Let $e = 4$ and $n = 4$. We have five graded Specht modules. For each standard tableau \mathbf{t} , we denote the tableau by its reading word: the *reading word* of \mathbf{t} is the permutation of $1, \dots, n$ obtained by reading the entries from left to right, starting with the first row and ending with the last row. We write v_{ijkl} for $v_{\mathbf{t}}$ when the reading word of \mathbf{t} is $ijkl$. The degree k part of S^λ is denoted by $(S^\lambda)_k$. By permuting letters, we have the right action of the symmetric group S_n on the set of tableaux.

As $S^{(2,2)}$ constitutes a semisimple block, we have $Y^{(2,2)} = S^{(2,2)} = D^{(2,2)}$. In particular, the decomposition matrix for this block is (1). For the grading, $S^{(2,2)} = (S^{(2,2)})_0 \oplus (S^{(2,2)})_1$, where $(S^{(2,2)})_0$ is spanned by v_{1324} and $(S^{(2,2)})_1$ is spanned by v_{1234} .

We consider the remaining four partitions. The action of t_1, t_2, t_3, t_4 and σ_1 are all zero on these graded Specht modules.

- $S^{(4)} = (S^{(4)})_1$ and $v_{1234} \in S^{(4)}e(0123)$. σ_2 and σ_3 act as zero.
- $S^{(3,1)} = (S^{(3,1)})_0 \oplus (S^{(3,1)})_1$, where $(S^{(3,1)})_0$ is spanned by $v_{1234} \in S^{(3,1)}e(0123)$, and $(S^{(3,1)})_1$ is spanned by $v_{1243} \in S^{(3,1)}e(0132)$ and $v_{1342} \in S^{(3,1)}e(0312)$. Hence, we have the matrix representation of the idempotents, with respect to the basis $(v_{1234}, v_{1243}, v_{1342})$, as follows. Note that the matrices act on row vectors from the right hand side.

$$e(0123) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e(0132) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } e(0312) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The action of σ_2 and σ_3 is given by

$$\sigma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- $S^{(2,1,1)} = (S^{(2,1,1)})_0 \oplus (S^{(2,1,1)})_1$, where $(S^{(2,1,1)})_0$ is spanned by $v_{1234} \in S^{(2,1,1)}e(0132)$ and $v_{1324} \in S^{(2,1,1)}e(0312)$, and $(S^{(2,1,1)})_1$ is spanned by $v_{1423} \in S^{(2,1,1)}e(0321)$. Hence, with respect to the basis $(v_{1234}, v_{1324}, v_{1423})$, we have

$$e(0132) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e(0312) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } e(0321) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The action of σ_2 and σ_3 is given by

$$\sigma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

- $S^{(1,1,1,1)} = (S^{(1,1,1,1)})_0$ and $v_{1234} \in S^{(1,1,1,1)}e(0321)$. σ_2 and σ_3 act as zero.

Thus, we know the following.

- (1) $S^{(1,1,1,1)} = D^{(1,1,1,1)}$ and $D^{(1,1,1,1)} = (D^{(1,1,1,1)})_0$.
- (2) $S^{(2,1,1)}$ contains a graded \mathcal{H}_4 -submodule

$$F(0, 0, 1) \simeq D^{(1,1,1,1)}[-1].$$

We write $D^{(2,1,1)} = S^{(2,1,1)} / D^{(1,1,1,1)}[-1]$. Then $D^{(2,1,1)} = (D^{(2,1,1)})_0$.

(3) $S^{(3,1)}$ contains a graded \mathcal{H}_4 -submodule

$$F(0, 1, 0) \oplus F(0, 0, 1) \simeq D^{(2,1,1)}[-1].$$

We write $D^{(3,1)} = S^{(3,1)} / D^{(2,1,1)}[-1]$. Then $D^{(3,1)} = (D^{(3,1)})_0$.

(4) $S^{(4)} \simeq D^{(3,1)}[-1]$.

If $\mu \neq (4)$, then μ is e -restricted and we may compute $d_{\lambda\mu}(v)$ by

$$d_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (S^\lambda : D^\mu[k]) v^k.$$

If $\mu = (4)$, $L(\mu)$ appears only in $W(\lambda)$ with $\lambda \supseteq \mu$ so that the only possibility is $d_{(4)(4)} = 1$. Hence, we have obtained the graded decomposition matrix. In the table, we write $d_{\lambda\mu}(v^{-1})$ instead of $d_{\lambda\mu}(v)$, in order to compare it with the Leclerc-Thibon canonical basis which we will compute below.

1^4		1
2,1^2		v 1
2^2		. . 1
3,1		. v . 1
4		. . . v 1

To phrase it in other terms, we have the following equations in the enriched Grothendieck group, in which we write the shift $[1]$ by v^{-1} .

$$\begin{aligned} [W((1, 1, 1, 1))] &= [L((1, 1, 1, 1))], \\ [W((2, 1, 1))] &= v[L((1, 1, 1, 1))] + [L((2, 1, 1))], \\ [W((2, 2))] &= [L((2, 2))], \\ [W((3, 1))] &= v[L((2, 1, 1))] + [L((3, 1))], \\ [W((4))] &= v[L((3, 1))] + [L((4))]. \end{aligned}$$

Hence, we have the following equalities in the dual space of the enriched Grothendieck group.

$$\begin{aligned} [L((1, 1, 1, 1))]^* &= v[W((2, 1, 1))]^* + [W((1, 1, 1, 1))]^*, \\ [L((2, 1, 1))]^* &= v[W((3, 1))]^* + [W((2, 1, 1))]^*, \\ [L((2, 2))]^* &= [W((2, 2))]^*, \\ [L((3, 1))]^* &= v[W((4))]^* + [W((3, 1))]^*, \\ [L((4))]^* &= [W((4))]^*. \end{aligned}$$

(Table 1)

We already know the decomposition matrix for the q -Schur algebra in the non-graded case. In the following table, the convention is the classical one, and the $(\lambda, \mu)^{th}$ entry is $d_{\lambda'\mu'}$. We confirm that it coincides with the specialization at $v = 1$ of the graded decomposition matrix.

```
gap> S:=Schur(4);
Schur(e=4, W(), P(), Pq())
gap> DecompositionMatrix(S,4);
```

$$\begin{array}{l|llll}
4 & & & & 1 \\
3,1 & & & 1 & 1 \\
2^2 & & & . & . & 1 \\
2,1^2 & & & . & 1 & . & 1 \\
1^4 & & & . & . & . & 1 & 1
\end{array}$$

We may also confirm the graded decomposition matrix by constructing non-split short exact sequences

$$0 \rightarrow S^\mu[k] \rightarrow M \rightarrow S^\lambda \rightarrow 0,$$

when $\text{Ext}^1(S^\lambda, S^\mu[k]) \neq 0$. The formula $d'_{\lambda\mu}(v) = \sum_{k \in \mathbb{Z}} (T(\mu') : W(\lambda')[k])v^{-k}$ implies that we must have

- $Y^{(1,1,1,1)} = S^{(1,1,1,1)}$,
- $Y^{(2,1,1)} = F_0 \supseteq F_1 = S^{(2,1,1)}$ and $F_0/F_1 = S^{(1,1,1,1)}[1]$,
- $Y^{(2,2)} = S^{(2,2)}$,
- $Y^{(3,1)} = F_0 \supseteq F_1 = S^{(3,1)}$ and $F_0/F_1 = S^{(2,1,1)}[1]$,
- $Y^{(4)} = F_0 \supseteq F_1 = S^{(4)}$ and $F_0/F_1 = S^{(3,1)}[1]$.

Let us consider $Y^{(2,1,1)}$, and compute the extension explicitly. The basis elements are

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1),$$

and the last three have degrees 0, 0, 1, respectively. Then, $Y^{(2,1,1)}$ is given by

$$\begin{aligned}
t_1 = 0, \quad t_2 = 0, \quad t_3 = 0, \quad t_4 &= \begin{pmatrix} 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
e(0132) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e(0312) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
e(0321) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\sigma_1 = 0, \quad \sigma_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

for any $\lambda \in F^\times$. Then,

$$(1, 0, 0, 0)\sigma_3 = \lambda(0, 0, 1, 0) \quad \text{or} \quad (1, 0, 0, 0)t_4 = -\lambda(0, 0, 0, 1)$$

determines the degree of $(1, 0, 0, 0)$ to be -1 .

We turn to the Leclerc-Thibon canonical basis. We denote them by $G(\mu)$. If $\mu \neq (4)$, we may compute them by the LLT algorithm. If $\mu = (4)$ then we

have $G(\mu) = s_\mu$ as μ has only one part. Thus, the canonical basis elements are given as follows.

$$\begin{aligned}
 G((1, 1, 1, 1)) &= vs_{(2,1,1)} + s_{(1,1,1,1)} (= f_1 f_2 f_3 f_0 v_{\Lambda_0}), \\
 G((2, 1, 1)) &= vs_{(3,1)} + s_{(2,1,1)} (= f_2 f_1 f_3 f_0 v_{\Lambda_0}), \\
 G((2, 2)) &= s_{(2,2)}, \\
 G((3, 1)) &= vs_{(4)} + s_{(3,1)} (= f_3 f_2 f_1 f_0 v_{\Lambda_0}), \\
 G((4)) &= s_{(4)}.
 \end{aligned}$$

(Table 2)

Comparing (Table 1) and (Table 2), we confirm that the coefficient matrices are identical. This example is rather an example for the Hecke algebra than an example for the q -Schur algebra, as we did not do any substantial calculation for the partitions which are not e -restricted. An interested reader may try larger size examples.

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⁸In reviewer’s remark in Mathematical Reviews, it is said “all the minus signs are missing” due to the fault of the publisher. Hence, I recommend reading arXiv:math/9902006.